

Hamilton's Harnack inequality and the W -entropy formula on complete Riemannian manifolds

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Abstract. In this paper, we prove Hamilton's Harnack inequality and Hamilton type gradient estimates of the logarithmic heat kernel for the Witten Laplacian on complete Riemannian manifolds. As application, we prove the W -entropy formula for the Witten Laplacian on complete Riemannian manifolds with natural Bakry-Emery Ricci curvature condition. We also prove a family of logarithmic Sobolev inequalities on complete Riemannian manifolds with μ -bounded geometry condition.

1 Introduction

In a seminal paper [41], Perelman proved the W -entropy formula for the Ricci flow on compact manifolds. As a consequence, the W -entropy of the Ricci flow is nondecreasing in time and the monotonicity is strict except that M is a shrinking Ricci solitons, i.e., for some $f \in C^\infty(M)$ and $\tau > 0$, it holds

$$Ric + \nabla^2 f = \frac{g}{2\tau}.$$

As was pointed out in [41], this result “removes the major stumbling block in Hamilton's approach to geometrization”. See also [8, 10, 21, 35].

During recent years, many people have established the W -entropy formula for various geometric evolution equations [9, 28, 29, 39, 40, 14, 31, 23]. In [39, 40], Ni proved an analogue of Perelman's W -entropy formula for the linear heat equation associated with the Laplace-Beltrami operator on complete Riemannian manifolds with fixed Riemannian metrics. More precisely, let M be an n -dimensional complete Riemannian manifold with a fixed Riemannian metric g , and let $u(x, \tau) = \frac{e^{-f(x, \tau)}}{(4\pi\tau)^{n/2}}$ be the heat kernel to the heat equation

$$\partial_\tau u = \Delta u.$$

The W -entropy for the above heat equation is defined by

$$W(u, \tau) = \int_M [\tau |\nabla f|^2 + f - n] \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv.$$

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Under some technical assumptions, Ni [39, 40] proved the following entropy formula

$$\frac{dW(u, \tau)}{d\tau} = -2 \int_M \tau \left(\left| \nabla^2 f - \frac{g}{2\tau} \right|^2 + Ric(\nabla f, \nabla f) \right) u dv,$$

which implies that $W(u, \tau)$ is decreasing in τ on complete Riemannian manifolds with non-negative Ricci curvature. See also [11].

In our previous paper [28], the author further extended the W -entropy formula to the heat equation associated with the Witten Laplacian on complete Riemannian manifolds. More precisely, let M be a complete Riemannian manifold with a fixed Riemannian metric g , $\phi \in C^2(M)$ and $d\mu = e^{-\phi} dv$, where v is the Riemannian volume measure on (M, g) , and let

$$L = \Delta - \nabla \phi \cdot \nabla$$

be the Witten Laplacian on (M, g) with respect to the weighted volume measure μ . Following [5, 32, 25], we introduce the m -dimensional Bakry-Emery Ricci curvature on (M, g, ϕ) by

$$Ric_{m,n}(L) := Ric + \nabla^2 \phi - \frac{\nabla \phi \otimes \nabla \phi}{m-n},$$

where $m \geq n$ is a constant, and $m = n$ if and only if ϕ is a constant. Let u be a positive solution of the heat equation

$$\partial_t u = Lu,$$

and define

$$H_m(u, t) = - \int_M u \log u d\mu - \frac{m}{2} (1 + \log(4\pi t)).$$

In [28], the author introduced the W -entropy for the Witten Laplacian by

$$W_m(u, t) = \frac{d}{dt} (t H_m(u, t)), \quad (1)$$

and proved that

$$W_m(u, t) = \int_M (t |\nabla f|^2 + f - m) \frac{e^{-f}}{(4\pi t)^{m/2}} d\mu, \quad (2)$$

where $u = \frac{e^{-f}}{(4\pi t)^{m/2}}$. Moreover, it was proved in [28] that the following W -entropy formula holds if $u = \frac{e^{-f}}{(4\pi t)^{m/2}}$ is the heat kernel of the Witten Laplacian on complete Riemannian manifolds satisfying the so-called μ -bounded geometry condition

$$\begin{aligned} \frac{dW_m(u, t)}{dt} &= -2 \int_M t \left(\left| \nabla^2 f - \frac{g}{2t} \right|^2 + Ric_{m,n}(L)(\nabla f, \nabla f) \right) u d\mu \\ &\quad - \frac{2}{m-n} \int_M t \left(\nabla \phi \cdot \nabla f + \frac{m-n}{2t} \right)^2 u d\mu. \end{aligned} \quad (3)$$

As a consequence, if (M, g, ϕ) satisfies the μ -bounded geometry condition and

$$Ric_{m,n}(L) \geq 0,$$

then the W -entropy is non-decreasing in time t , i.e.,

$$\frac{dW_m(u, t)}{dt} \leq 0, \quad \forall t \geq 0,$$

which extends the above result due to Ni [39, 40]. Moreover, under the above conditions, the author proved that $\frac{d}{dt}W_m(u, t) = 0$ for some $t > 0$ if and only if M is isometric to \mathbb{R}^n , $m = n$, ϕ is a constant, $L = \Delta$, and u is the standard Gaussian heat kernel on \mathbb{R}^n .

When $m \in \mathbb{N}$, consider the warped product metric on $M^n \times S^{m-n}$ defined by

$$\tilde{g} = g_M \oplus e^{-\frac{2\phi}{m-n}} g_{S^{m-n}}.$$

where S^{m-n} is the unit sphere in \mathbb{R}^{m-n+1} with the standard metric $g_{S^{m-n}}$. By a classical result in Riemannian geometry, the quantity $Ric_{m,n}(L)$ is equal to the Ricci curvature of the above warped product metric \tilde{g} on $M^n \times S^{m-n}$ along the horizontal vector fields. See [32, 25, 49] and reference therein. Thus, the m -dimensional Bakry-Emery Ricci curvature $Ric_{m,n}(L)$ has a very natural geometric interpretation. On the other hand, according to [28], we say that (M, g, ϕ) satisfies the μ -bounded geometry condition, if the Riemannian curvature tensor as well as its first and second order derivatives are uniformly bounded, $\phi \in C^4(M)$ with $\nabla \phi \in C_b^3(M)$, and the μ -volume of the geodesic balls of a fixed radius $r_0 > 0$ is uniformly bounded from below by a positive constant with respect to their centers, i.e.,

$$\mu_0 := \inf_{x \in M} \mu(B(x, r_0)) > 0. \quad (4)$$

It is easy to see that the uniformly volume lower bound condition (4) implies that the total μ -volume of M is infinity if M is non-compact. This excludes the case where μ is a probability measure on M . Moreover, by Theorem 2.8 in [27], on complete Riemannian manifolds with $Ric_{m,n}(L) \geq -K$ for some constant $K \geq 0$ and with the uniformly lower bound condition (4), the following Sobolev inequality holds

$$\|f\|_{\frac{2m}{m-2}}^2 \leq C \left(\int_M |\nabla f|^2 d\mu + \int_M |f|^2 d\mu \right), \quad \forall f \in C_0^\infty(M).$$

Thus, at least from a geometric and analytic point of view, the uniformly volume lower bound condition (4) is too restrictive for the validity of the W -entropy formula (3) for the Witten Laplacian on complete Riemannian manifolds. It is very natural to ask the question whether the W -entropy formula (3) remains valid on complete Riemannian manifolds without assuming the uniformly volume lower bound condition (4). This is certainly an important issue in view of an extensive interest on the study of geometry and analysis on complete Riemannian manifolds with smooth weighted volume measures [5, 6, 16, 25, 28, 29, 38, 49, 48, 36, 37].

The purpose of this paper is to prove the W -entropy formula for the Witten Laplacian on complete Riemannian manifolds with more natural geometric conditions in which one need not to assume the uniform lower bound condition (4). Indeed, after the paper [28] was accepted, we observed that the uniform volume lower bound condition (4) can be removed. See the Note added in proof of [28]. Our observation is essentially based on the fact that one can prove some Hamilton type gradient estimates for the logarithmic of the heat kernel of the Witten Laplacian on complete Riemannian manifolds on which one need not to assume the uniform volume lower bound condition (4). In [28], the author obtained these type gradient estimates on complete Riemannian manifolds satisfying the μ -bounded geometry condition. In this paper, we prove Hamilton's Harnack inequality for all bounded and positive solutions to the heat equation of the Witten Laplacian and prove the Hamilton type gradient estimates of the logarithmic of the heat kernel for the Witten Laplacian on complete Riemannian manifolds with natural geometric conditions. As application, we derive the W -entropy formula and prove the rigidity theorem of the W -entropy for the Witten Laplacian on complete Riemannian manifolds with natural Bakry-Emery curvature condition.

The rest of this paper is organized as follows. In Section 2, we state the main results of this paper. In Section 3, we prove Hamilton's Harnack inequality for the Witten Laplacian

on complete Riemannian manifolds. In Section 4, we prove the Hamilton type gradient estimates of the logarithmic heat kernel of the Witten Laplacian using a method of stochastic analysis. In Section 5, we prove the W -entropy formula and derive the monotonicity and rigidity theorem of the W -entropy for the Witten Laplacian on complete Riemannian manifolds with natural Bakry-Emery Ricci curvature condition. In Section 6, we prove a family of logarithmic Sobolev inequalities on complete Riemannian manifolds with μ -bounded geometry condition. In Section 7, we give an analytic proof of the first order gradient estimate of the logarithmic heat kernel of the Witten Laplacian operator. In Section 8 we develop another probabilistic approach to the Hamilton type Harnack inequality for the Witten Laplacian on complete Riemannian manifolds.

2 Statement of the main results

Let (M, g) be a complete Riemannian manifold, $f \in C^2(M)$ and $d\mu = e^{-f} dv$, where dv denotes the Riemannian volume measure on (M, g) . For all $u, v \in C_0^\infty(M)$, the following integration by parts formula holds

$$\int_M \langle \nabla u, \nabla v \rangle d\mu = - \int_M Luv d\mu = - \int_M uLv d\mu,$$

where $L = \Delta - \nabla \phi \cdot \nabla$ is the Witten Laplacian on (M, g, ϕ) . In [5], Bakry and Emery proved that for all $u \in C_0^\infty(M)$,

$$L|\nabla u|^2 - 2\langle \nabla u, \nabla Lu \rangle = 2|\nabla^2 u|^2 + 2(Ric + \nabla^2 \phi)(\nabla u, \nabla u). \quad (5)$$

The formula (5) can be viewed as a natural extension of the Bochner-Weitzenböck formula. The quantity $Ric + \nabla^2 \phi$, which is called in the literature the Bakry-Emery Ricci curvature on the weighted Riemannian manifolds (M, g, ϕ) , and plays as a good substitute of the Ricci curvature in many problems in comparison geometry and analysis on complete Riemannian manifolds with smooth weighted volume measures. See [5, 16, 25, 28, 29, 38, 49, 36, 37] and reference therein.

Consider the heat equation

$$\frac{\partial u}{\partial t} = Lu. \quad (6)$$

The heat kernel $p_t(x, y)$ to the heat equation (6) is defined by

$$u(t, x) = \int_M u(0, y) p_t(x, y) d\mu(y).$$

We now state the main results of this paper. The first result of this paper is the following Hamilton's Harnack inequality for the heat equation of the Witten Laplacian on complete Riemannian manifolds.

Theorem 2.1 *Let M be a complete Riemannian manifold, and $\phi \in C^2(M)$. Suppose that there exists a constant $K \geq 0$ such that*

$$Ric + \nabla^2 \phi \geq -K.$$

Let u be a positive and bounded solution to the heat equation

$$\partial_t u = Lu.$$

Then for all $x \in M$ and $t > 0$,

$$|\nabla \log u|^2 \leq \left(\frac{1}{t} + 2K \right) \log(A/u), \quad (7)$$

where

$$A := \sup\{u(t, x) : x \in M, t \geq 0\}.$$

As a corollary of Theorem 2.1, we can derive the following gradient estimate and the Liouville theorem due to Brighton [7]. See also Remark 3.4 below.

Corollary 2.2 *Let u be a bounded and positive L -harmonic function u on M . Then, under the same condition as in Theorem 2.1, we have*

$$|\nabla \log u| \leq 2K \log(A/(u - \inf u)),$$

where $A = \sup u - \inf u$. In particular, if $\text{Ric} + \nabla^2 \phi \geq 0$, then every bounded L -harmonic function must be constant.

The following two results provide us with the Hamilton type gradient estimates of the logarithmic heat kernel of the Witten Laplacian on complete Riemannian manifolds with natural geometric conditions.

Theorem 2.3 *Let M be a complete Riemannian manifold, and $\phi \in C^2(M)$. Suppose that there exist some constant $m \geq n$ and $K \geq 0$ such that*

$$\text{Ric}_{m,n}(L) \geq -K.$$

Then, for all $T > 0$, there exists a constant $C(K, m, n, T) > 0$ such that

$$|\nabla \log p_t(x, y)| \leq C(K, m, n, T) \left(\frac{d(x, y)}{t} + \frac{1}{\sqrt{t}} \right)$$

holds for all $x, y \in M$ and $t \in (0, T]$.

Theorem 2.4 *Let (M, g) be an n -dimensional complete Riemannian manifold. Suppose that $\|\nabla^k \text{Riem}\| \leq C_k$ for some constant $C_k > 0$, $0 \leq k \leq N$, $\phi \in C^{N+1}(M)$ with $\nabla \phi \in C_b^N(M)$, and there exists some $m \geq n$, $K \geq 0$ such that $\text{Ric}_{m,n}(L) \geq -K$. Then, for all $T > 0$, there exists a constant $C_N(K, m, n, T) > 0$ such that*

$$|\nabla^N \log p_t(x, y)| \leq C_N(K, m, n, T) \left(\frac{d(x, y)}{t} + \frac{1}{\sqrt{t}} \right)^N$$

holds for all $x, y \in M$ and $t \in (0, T]$.

Based on Theorem 2.3 and Theorem 2.4, we have the following result on the W -entropy for the Witten Laplacian on complete Riemannian manifold, which improves a previous result obtained in [28].

Theorem 2.5 *Let (M, g) be a complete Riemannian manifold. Suppose that $\|\nabla^k \text{Riem}\| \leq C_k$ for some constant $C_k > 0$, $0 \leq k \leq 2$, $\phi \in C^4(M)$ with $\nabla \phi \in C_b^3(M)$, and there exists some $m \geq n$, $K \geq 0$ such that $\text{Ric}_{m,n}(L) \geq -K$. Let*

$$u(t, x) = \frac{e^{-f}}{(4\pi t)^{m/2}}.$$

be the heat kernel of the heat equation

$$\frac{\partial u}{\partial t} = Lu.$$

Then, for all $t > 0$, we have the following entropy formula

$$\begin{aligned} \frac{dW_m(u, t)}{dt} &= -2 \int_M t \left(\left| \nabla^2 f - \frac{g}{2\tau} \right|^2 + Ric_{m,n}(L)(\nabla f, \nabla f) \right) u d\mu \\ &\quad - \frac{2}{m-n} \int_M t \left(\nabla \phi \cdot \nabla f + \frac{m-n}{2t} \right)^2 u d\mu, \end{aligned}$$

where

$$W_m(u, t) = \int_M (t|\nabla f|^2 + f - m) \frac{e^{-f}}{(4\pi t)^{m/2}} d\mu$$

is the W -entropy of the Witten Laplacian $L = \Delta - \nabla \phi \cdot \nabla$ on (M, g, μ) .

As a consequence of the above W -entropy formula, we have the following monotonicity and rigidity theorem for the W -entropy for the Witten Laplacian on complete Riemannian manifolds, which also improves the previous result obtained in [28].

Theorem 2.6 *Let (M, g) be a complete Riemannian manifold. Suppose that $\|\nabla^k \text{Riem}\| \leq C_k$ for some constant $C_k > 0$, $0 \leq k \leq 2$, $\phi \in C^4(M)$ with $\nabla \phi \in C_b^3(M)$. Suppose that there exists a constant $m \geq n$ such that*

$$Ric_{m,n}(L) \geq 0.$$

Then

$$\frac{dW_m(u, t)}{dt} \leq 0, \quad \forall t \geq 0.$$

That is, $W_m(u, t)$ is non-increasing in time t . Moreover, $W_m(u, t)$ attains its minimum at some point $t_0 > 0$, i.e.,

$$\frac{dW_m(u, t)}{dt} = 0$$

holds at $t = t_0$, if and only if (M, g) is isometric to Euclidean space \mathbb{R}^n , $m = n$, $\phi \equiv C$ for a constant $C \in \mathbb{R}$, and

$$u(t, x) = \frac{e^{-\frac{\|x\|^2}{4t}}}{(4\pi t)^{n/2}}, \quad \forall x \in \mathbb{R}^n, t > 0.$$

The following result says that the uniform μ -volume lower bound condition (4) is indeed an important condition which implies the validity of a family of logarithmic Sobolev inequalities on complete Riemannian manifolds with μ -bounded geometry condition.

Theorem 2.7 *Let M be a complete Riemannian manifold, and $\phi \in C^2(M)$. Suppose that there exist some constants $m \geq n$ and $K \geq 0$ such that*

$$Ric_{m,n}(L) \geq -K,$$

and the uniform volume lower bound condition (4) holds, i.e., there exists a constant $r_0 > 0$ such that

$$\mu_0 = \inf\{\mu(B(x, r_0)) : x \in M\} > 0.$$

Then, for any $\tau > 0$, the following logarithmic Sobolev inequality holds: for all $v \in W^{1,2}(M, \mu)$ with $\int_M v^2 d\mu = 1$, we have

$$\int_M v^2 \log v^2 d\mu \leq 4\tau \int_M |\nabla v|^2 - m \left(1 + \frac{1}{2} \log(4\pi\tau) \right) - \mu(\tau),$$

where

$$\mu(\tau) := \inf \left\{ W_m(u, \tau) : u = \frac{e^{-f}}{(4\pi\tau)^{m/2}}, \int_M u d\mu = 1 \right\}$$

is a finite number.

Remark 2.8 When $m \in \mathbb{N}$, as was pointed out in [28], the quantity

$$H_m(u, t) = - \int_M u \log u d\mu - \frac{m}{2} (1 + \log(4\pi t))$$

defined in (1) is the difference of the Boltzmann-Shannon-Nash entropy

$$H(u, t) = - \int_M u \log u d\mu$$

of the time t -heat kernel measure $u d\mu$ on (M, g) and the Boltzmann-Shannon-Nash entropy

$$H(\gamma_m, t) = \frac{m}{2} (1 + \log(4\pi t))$$

of the time t -Gaussian measure $\gamma_m(\cdot, t)$ on \mathbb{R}^m , where

$$d\gamma_m(x, t) = \frac{e^{-\frac{\|x\|^2}{4t}}}{(4\pi t)^{m/2}} dx.$$

This observation together with the definition formula (1), i.e.,

$$W_m(u, t) = \frac{d}{dt}(tH_m(u, t)),$$

gives a probabilistic interpretation of the W -entropy functional (2) for the Witten Laplacian on complete Riemannian manifolds. See also [28] for the probabilistic interpretation of Perelman's W -entropy for the Ricci flow.

Remark 2.9 Theorem 2.3 and Theorem 2.4 are improved versions of Theorem 4.1 and Theorem 4.2 due to the author [28], where one need to assume the additional condition (4) holds. When M is a compact Riemannian manifold and ϕ is a constant, Theorem 2.3 and Theorem 2.4 are due to Hamilton [18], Sheu [42], Hsu [20] and Stroock-Turesky [44]. In the case where $L = \Delta$ is the Laplace-Beltrami operator on complete Riemannian manifolds with bounded geometry condition (in particular, the uniform volume lower bound condition (4) holds for $\mu = \nu$) and for small time $t > 0$, Theorem 2.3 is due to Engoulatov [15]. In [22, 13], the authors proved the gradient estimates in Theorem 2.3 for $L = \Delta$ on complete Riemannian manifolds with non-negative Ricci curvature.

Remark 2.10 In a very recent paper [30], Songzi Li and the author gave a new proof of the W -entropy formula on complete Riemannian manifolds by using the warped metric product

$$\tilde{g} = g_M \bigoplus e^{-\frac{2\phi}{m-n}} g_N.$$

on $M \times N$, where (N, g_N) is any fixed compact Riemannian manifold of dimension $m - n$. This gives a natural geometric interpretation of the last term appeared in (3). See Remark 2.2 in [30]. Moreover, we extended the W -entropy formula to the heat equation of the time dependent Witten Laplacian on compact Riemannian manifolds on which the metrics $\{g(t), t \in [0, T]\}$ and the potential functions $\{\phi(t), t \in [0, T]\}$ evolve along the m -dimensional Perelman Ricci flow and the conjugate heat equation

$$\begin{aligned}\frac{\partial g}{\partial t} &= -2 \left(Ric + \nabla^2 \phi - \frac{\nabla \phi \otimes \nabla \phi}{m - n} \right), \\ \frac{\partial \phi}{\partial t} &= -R - \Delta \phi + \frac{|\nabla \phi|^2}{m - n}.\end{aligned}$$

In this case, without assuming $Ric_{m,n}(L) \geq 0$, we prove in [30] that the W -entropy for the positive solution of the forward heat equation

$$\partial_t u = Lu$$

of the time dependent Witten Laplacian $L = \Delta - \nabla \phi \cdot \nabla$, is non-increasing in time $t \in [0, T]$.

3 Hamilton's Harnack inequality

In this section we prove Hamilton's Harnack inequality for the Witten Laplacian.

Theorem 3.1 (*i.e., Theorem 2.1*) *Let u be a bounded and positive solution of the heat equation $\partial_t u = Lu$. Suppose that $Ric + \nabla^2 \phi \geq -K$. Then for all $T > 0$, we have*

$$|\nabla \log u(T, x)|^2 \leq \left(\frac{1}{T} + 2K \right) \log \left[\frac{A}{u(T, x)} \right],$$

where

$$A := \sup\{u(t, x) : x \in M, t \in [0, T]\}.$$

Proof. In compact case, we can prove it by the same argument as used by Hamilton [18] for the Laplace-Beltrami. Indeed, a direct calculation yields

$$\begin{aligned}(\partial_t - L) \frac{|\nabla u|^2}{u} &= -\frac{2}{u} |\nabla^2 u - u^{-1} \nabla u \otimes \nabla u|^2 - 2u^{-1} (Ric + \nabla^2 \phi)(\nabla u, \nabla u) \\ &\leq 2Ku^{-1} |\nabla u|^2.\end{aligned}$$

Moreover

$$(\partial_t - L)(u \log(A/u)) = \frac{|\nabla u|^2}{u}.$$

Let $\phi(t) = \frac{t}{2Kt+1}$. Then

$$\phi'(t) + 2K\phi(t) \leq 1.$$

Define

$$h(x, t) := \phi(t) \frac{|\nabla u|^2}{u} - u \log(A/u).$$

We have

$$\begin{aligned}
(\partial_t - L)h &= \phi'(t) \frac{|\nabla u|^2}{u} + \phi(t)(\partial_t - L) \frac{|\nabla u|^2}{u} - (\partial_t - L)u \log(A/u) \\
&\leq \phi'(t) \frac{|\nabla u|^2}{u} + 2K\phi(t) \frac{|\nabla u|^2}{u} - \frac{|\nabla u|^2}{u} \\
&= (\phi'(t) + 2K\phi(t) - 1) \frac{|\nabla u|^2}{u} \\
&\leq 0.
\end{aligned}$$

Note that

$$h(x, 0) = -u \log(A/u)(x, 0) \leq 0.$$

By the maximum principle, we have $h(x, t) \leq 0$ for all time $t > 0$ and $x \in M$. This proves Theorem 3.1 in compact case. In general case of complete non-compact Riemannian manifolds with $Ric(L) \geq -K$, we can give a probabilistic proof to Hamilton's Harnack inequality as follows. Indeed, on any complete Riemannian manifold M with $Ric + \nabla^2 \phi \geq K$, a previous result due to Bakry [3] says that the L -diffusion process on M has infinity lifetime. Let X_t be the L -diffusion process on M starting from $X_0 = x$. Applying Itô's formula to $h(X_t, T - t)$, $t \in [0, T]$, we have

$$h(X_t, T - t) = h(X_0, T) + \int_0^t \nabla h(X_s, T - s) \cdot dW_s + \int_0^t \left(L - \frac{\partial}{\partial t} \right) h(X_s, T - s) ds,$$

where the second term in the right hand side is the Itô's stochastic integral with respect to a M -valued Brownian motion $\{W_s, s \in [0, t]\}$. In particular, taking $t = T$, we obtain

$$h(X_T, 0) = h(X_0, T) + \int_0^T \nabla h(X_s, T - s) \cdot dW_s + \int_0^T \left(L - \frac{\partial}{\partial t} \right) h(X_s, T - s) ds.$$

Taking the expectation on both sides, the martingale property of Itô's integral implies that

$$E[h(X_T, 0)] = h(x, T) + E \left[\int_0^T \left(L - \frac{\partial}{\partial t} \right) h(X_s, T - s) ds \right] \geq h(x, T).$$

As $h(y, 0) \leq 0$ for all $y \in M$, we derive that $h(x, T) \leq 0$ for all $T > 0$ and $x \in M$. This finishes the proof of Theorem 3.1 on complete non-compact Riemannian manifolds. \square

An alternative proof of Theorem 3.1. By [4, 6, 48], we know that $Ric(L) = Ric + \nabla^2 \phi \geq -K$ holds if and only the following version of logarithmic Sobolev inequality holds: for all $T > 0$, $f \in C_b(M)$ with $f > 0$,

$$\frac{|\nabla P_T f|^2}{P_T f} \leq \frac{2K}{1 - e^{-2KT}} (P_T(f \log f) - P_T f \log P_T f).$$

Replacing f by $P_t f$ and using the fact that $0 < P_t f(x) \leq A$ for all $x \in M$, one has

$$\frac{|\nabla P_{T+t} f|^2}{P_{T+t} f} \leq \frac{2K}{1 - e^{-2KT}} (P_T(P_t f \log A) - P_{T+t} f \log P_{T+t} f).$$

Thus

$$|\nabla \log P_{T+t} f|^2 \leq \frac{2K}{1 - e^{-2KT}} \log(A/P_{T+t} f), \quad \forall t > 0.$$

Taking $t \rightarrow 0^+$ we have

$$\begin{aligned} |\nabla \log P_T f|^2 &\leq \frac{2K}{1 - e^{-2KT}} \log(A/P_T f) \\ &\leq 2K \left(1 + \frac{1}{2KT}\right) \log(A/P_T f) \\ &\leq \left(2K + \frac{1}{T}\right) \log(A/P_T f). \end{aligned}$$

The proof of Theorem 3.1 is completed. \square

Remark 3.2 As was pointed out in Section 1, Theorem 2.1 was first proved by Hamilton [18] for the heat equation of the Laplace-Beltrami operator on compact Riemannian manifolds and was extended to complete Riemannian manifolds with Ricci curvature bounded from below by Kotschwar [22]. For a probabilistic proof of Hamilton's Harnack inequality for $L = \Delta$ on compact or complete Riemannian manifolds, see [1]. The local version of Theorem 2.1 for the Laplace-Beltrami operator was proved by Souplet-Zhang [43]. In [2] (Theorem 5.1), Arnaudon, Tahlmaier and Wang extended the local estimate of Souplet-Zhang [43] to the heat equation of the Laplacian with general drift $L = \Delta + Z$, where Z is not necessary of gradient form $Z = \nabla \phi$ and the Laplacian comparison theorem for $L = \Delta + Z$ is essentially used. Taking $R \rightarrow \infty$ in [2], we can derive the following estimate for the bounded and positive solution of the heat equation $\partial_t u = Lu$: there exists a constant $C > 0$, which depends only on $n = \dim M$, such that for all $x \in M$ and $t > 0$, it holds

$$|\nabla \log u|^2 \leq C \left(\frac{1}{t} + K \right) [1 + \log(A/u)]. \quad (8)$$

where $A := \sup\{u(t, x) : x \in M, t \geq 0\}$. Note that the Harnack inequality (7) is stronger than (8). See also Section 8 for another probabilistic proof of the Hamilton type Harnack inequality for the Witten Laplacian on complete Riemannian manifolds.

As a corollary of the above theorem, we can derive the following gradient estimate and the Liouville theorem due to Brighton [7] for bounded L -harmonic functions.

Corollary 3.3 (*i.e., Corollary 2.2*) Suppose that $\text{Ric}(L) = \text{Ric} + \nabla^2 \phi \geq -K$ for some constant $K \geq 0$. Then every bounded L -harmonic function u satisfies

$$|\nabla \log u| \leq 2K \log(A/(u - \inf u)), \quad (9)$$

where $A = \sup u - \inf u$. In particular, if $\text{Ric}(L) = \text{Ric} + \nabla^2 \phi \geq 0$, then every bounded L -harmonic function must be constant.

Proof. Applying (8) to $u(t, x) = u(x) - \inf u$, $x \in M$, $t > 0$, we have

$$|\nabla \log u(x)|^2 \leq \left(2K + \frac{1}{T}\right) \log(A/(u(x) - \inf u)), \quad \forall T > 0.$$

Taking $T \rightarrow \infty$, we derive (9). In particular, if $K = 0$, we have $|\nabla \log u| = 0$. This finishes the proof of Corollary 3.3. \square

Remark 3.4 In the case $L = \Delta$, Yau [46] proved that, on complete Riemannian manifolds with Ricci curvature bounded below by $-K$, i.e., $\text{Ric} \geq -K$, every harmonic function which is bounded below by a constant satisfies the gradient estimate

$$|\nabla \log u| \leq \sqrt{(n-1)K}(u - \inf u).$$

In particular, Yau [46] proved the Strong Liouville theorem holds: On complete Riemannian manifolds with non-negative Ricci curvature, every positive (and hence bounded) harmonic function must be constant. See also [45]. In [25], the author extended Yau's gradient estimate and Strong Liouville theorem to the L -harmonic functions on complete Riemannian manifolds via the Bakry-Emery Ricci curvature: Suppose that $Ric_{m,n}(L) \geq -K$ for some constant $K \geq 0$. Then every L -harmonic function u , which is bounded from below, satisfies the following gradient estimate

$$|\nabla \log u| \leq \sqrt{(m-1)K}(u - \inf u).$$

In particular, if $Ric_{m,n}(L) \geq 0$, then all positive (and hence bounded) L -harmonic functions must be constant. In [26], the local gradient estimates was proved for L -harmonic functions: If $Ric_{m,n}(L) \geq -K$, then every positive L -harmonic function u on M satisfies: for all $o \in M$, and $R > 0$, we have

$$|\nabla \log u(x)| \leq \sqrt{(m-1)K}u(x) + C(K, m)(R^{-1/2} + R^{-3/2}), \quad \forall x \in B(o, R).$$

In [7], Brighton proved that, on complete Riemannian manifolds with $Ric + \nabla^2 \phi \geq 0$, every bounded L -harmonic function must be constant, i.e., Corollary 2.2. Our method for the proof of Corollary 2.2 is different from [7]. For more recent results, see [36, 37].

4 Gradient estimates of the heat kernel

In this section we use a probabilistic approach to prove Theorem 2.3 and Theorem 2.4. Our proof is inspired by previous works initiated by Sheu [42], and developed by Hsu [20] and Engoulatov [15]. An alternative proof of Theorem 2.3 is given in Section 8 below.

Let $O(M)$ be the orthogonal frame bundle over M , $\pi : O(M) \rightarrow M$ the canonical projection map. Let H_1, \dots, H_n be the canonical horizontal vector fields on $O(M)$. Let B_t be the standard Brownian motion on \mathbb{R}^n . Following Malliavin [33, 34], we define the horizontal L -diffusion process U_t on $O(M)$ by the following Stratonovich SDE on $O(M)$:

$$dU_t = \sum_{i=1}^n H_i(U_t) \circ dB_t^i - \nabla^H \phi(U_t) dt, \quad (10)$$

where $\nabla^H \phi$ denotes the horizontal lift of the gradient vector field $\nabla \phi$ on $O(M)$, which is the unique horizontal vector field on $O(M)$ such that

$$\pi_*(\nabla^H \phi) = \nabla \phi.$$

Let

$$X_t = \pi(U_t), \quad \forall t > 0.$$

Then X_t is a diffusion process on M with infinitesimal generator L . Moreover, we have

$$dX_t = U_t \circ dB_t - \nabla \phi(X_t) dt. \quad (11)$$

Let $\Delta_{O(M)} = \sum_{i=1}^n H_i^2$ be the horizontal Laplace-Beltrami on $O(M)$. The horizontal Witten Laplacian is defined by

$$L_{O(M)} = \Delta_{O(M)} - \nabla^H \phi \cdot \nabla^H.$$

Fix $T > 0$. Let $\mathbb{P}_{x,y,T}$ be the conditional L -diffusion processes starting from x and ending at y at time T , i.e.,

$$\mathbb{P}_{x,y,T}(\cdot) = \mathbb{P}_x(\cdot | X_T = y).$$

Let

$$J(t, u) = \log p_{T-t}(\pi u, y), \quad \forall u \in O(M), t \in [0, T].$$

By the Girsanov transformation, we have

$$\left. \frac{d\mathbb{P}_{x,y,T}}{d\mathbb{P}_x} \right|_{\mathcal{F}_t} = \frac{p_{T-t}(X_t, y)}{p_T(x, y)} = \exp\{J(t, U_t) - J(0, U_0)\}, \quad \forall t < T.$$

where $\mathcal{F}_t = \sigma(X_s, s \leq t)$ is the natural σ -field generated by X_t . By Itô's formula, we have

$$dJ(t, U_t) = \frac{\partial J}{\partial t}(t, U_t)dt + \sqrt{2}\langle \nabla^H J(t, U_t), dB_t \rangle + L_{O(M)}J(t, U_t)dt,$$

Note that $J(t, u)$ satisfies the Hamilton-Jacobi equation

$$\frac{\partial J}{\partial t} + L_{O(M)}J + |\nabla^H J|^2 = 0, \quad (12)$$

which yields

$$\left. \frac{d\mathbb{P}_{x,y,T}}{d\mathbb{P}_x} \right|_{\mathcal{F}_t} = \exp \left[\int_0^t \langle \sqrt{2}\nabla^H J(s, U_s), dB_s \rangle - \int_0^t |\nabla^H J(s, U_s)|^2 ds \right], \quad \forall t < T.$$

It follows, by Girsanov's theorem, that the process

$$b_t = B_t - \sqrt{2} \int_0^t \nabla^H J(s, U_s) ds, \quad s \in [0, T], \quad (13)$$

is a Brownian motion under $\mathbb{P}_{x,y,T}$. Substituting (13) into (10) and (11), we have the following lemma.

Lemma 4.1 *The conditional L -diffusion process X_t on M starting from x with terminal end y at the time T , and its horizontal lift U_t , are the solutions of the following Stratonovich SDEs:*

$$\begin{aligned} dU_t &= \sum_{i=1}^n H_i(U_t) \circ \left[\sqrt{2}db_t^i + 2H_i \log p_{T-t}(U_t, y)dt \right] - \nabla^H \phi(U_t)dt, \\ dX_t &= U_t \circ \left[\sqrt{2}db_t + 2\nabla^H \log p_{T-t}(U_t, y)dt \right] - \nabla \phi(X_t)dt, \end{aligned}$$

where b_t is a standard Brownian motion on \mathbb{R}^n under $\mathbb{P}_{x,y,T}$.

Proposition 4.2 *For all $0 < t < T$, we have*

$$\mathbb{E}[d|\nabla^H J(t, U_t)|^2] = 2\mathbb{E}[|\nabla^H \nabla^H J|^2 + (\text{Ric} + \nabla^2 \phi)(\nabla^H J, \nabla^H J)]. \quad (14)$$

Proof. By Lemma 4.1 and Itô's formula, we have

$$d|\nabla^H J(t, U_t)|^2 = \langle \nabla^H |\nabla^H J|^2, \sqrt{2}db_t + 2\nabla^H J dt \rangle + (\partial_t + L_{O(M)})|\nabla^H J|^2 dt.$$

By the generalized Bochner-Weitzenböck formula, the action of the horizontal Witten Laplacian $L_{O(M)}$ on $|\nabla^H J|^2$ is given by

$$L_{O(M)}|\nabla^H J|^2 = 2\langle \nabla^H L_{O(M)} J, \nabla^H J \rangle + 2|\nabla^H \nabla^H J|^2 + 2(Ric + \nabla^2 \phi)(\nabla^H J, \nabla^H J).$$

On the other hand, using the Hamilton-Jacobi equation (12), we have

$$\partial_t |\nabla^H J|^2 = 2\langle \partial_t \nabla^H J, \nabla^H J \rangle = -2\langle L_{O(M)} \nabla^H J, \nabla^H J \rangle - 2\langle \nabla^H |\nabla^H J|^2, \nabla^H J \rangle.$$

Combining this with the previous calculation, we have

$$d|\nabla^H J(t, U_t)|^2 = \langle \nabla^H |\nabla^H J|^2, \sqrt{2}db_t \rangle + 2[|\nabla^H \nabla^H J|^2 + (Ric + \nabla^2 \phi)(\nabla^H J, \nabla^H J)] dt.$$

Taking expectation with respect to $\mathbb{E}_{x,y,T}$, we complete the proof of Proposition 4.2. \square

Proposition 4.3 *Suppose that $Ric + \nabla^2 \phi \geq -K$. Then for all $T > 0$, we have*

$$|\nabla \log p_T(x, y)|^2 \leq 2 \left(\frac{1}{T} + K \right) \mathbb{E} \left[\log \frac{p_{T/2}(X_{T/2}, y)}{p_T(x, y)} \right]. \quad (15)$$

Proof. Based on Proposition 4.2, the proof of Proposition 4.3 is similar to the argument used in [20, 15]. For the convenience of the reader, we give the detail here. Under the condition $Ric + \nabla^2 \phi \geq -K$, integrating (14) from 0 to t , we have

$$\mathbb{E}[|\nabla^H J(t, U_t)|^2] - |\nabla^H J(0, U_0)|^2 \geq -2KE \left[\int_0^t |\nabla^H J(s, U_s)|^2 ds \right].$$

Integrating again with respect to t we can obtain

$$\frac{T}{2} |\nabla^H J(0, U(0))|^2 \leq (1 + KT) \mathbb{E} \left[\int_0^{T/2} |\nabla^H J(s, U_s)|^2 ds \right]. \quad (16)$$

By Itô's formula and the Hamilton-Jacobi equation (12), we have

$$\begin{aligned} dJ(t, U_t) &= (\partial_t + L_{O(M)})J dt + \langle \sqrt{2} \nabla^H J, db_t + \sqrt{2} \nabla^H J dt \rangle \\ &= \langle \sqrt{2} \nabla^H J, db_t \rangle + |\nabla^H J|^2 dt. \end{aligned}$$

Taking expectation with respect to $\mathbb{E}_{x,y,T}$ and integrating from $t = 0$ to $t = T/2$, we get

$$\mathbb{E}[J(T/2, U_{T/2})] - J(0, U_0) = \mathbb{E} \left[\int_0^{T/2} |\nabla^H J|^2 dt \right]. \quad (17)$$

Combining (16) with (17), we finish the proof of (15). \square

Proposition 4.4 *Suppose that there exist some constants $m \geq n$, $m \in \mathbb{N}$ and $K \geq 0$ such that $Ric_{m,n}(L) \geq -K$. Then, for any small $\varepsilon > 0$, there exist some constants $C_i = C_i(m, n, K, \varepsilon) > 0$, $i = 1, 2$, such that for all $x, y \in M$ and $t > 0$,*

$$\begin{aligned} p_t(x, y) &\leq \frac{C_1}{\mu(B_y(\sqrt{t}))} \exp \left(-\frac{d^2(x, y)}{4(1 + \varepsilon)t} + \alpha \varepsilon K t \right) \\ &\quad \times \left(\frac{d(x, y) + \sqrt{t}}{\sqrt{t}} \right)^{m/2} \exp \left(\frac{\sqrt{(m-1)K} d(x, y)}{2} \right), \end{aligned} \quad (18)$$

where α is a constant depending only on m , and

$$p_t(x, y) \geq C_2 e^{-(1+\varepsilon)\lambda_{K,m}t} \mu^{-1}(B_y(\sqrt{t})) \exp\left(-\frac{d^2(x, y)}{4(1-\varepsilon)t}\right) \left[\frac{\sqrt{K}d(x, y)}{\sinh \sqrt{K}d(x, y)}\right]^{\frac{m-1}{2}}, \quad (19)$$

where

$$\lambda_{K,m} = \frac{(m-1)^2 K}{8}.$$

Proof. The lower bound estimate (19) has proved in [28]. By Theorem 5.4 in [25], for all $\varepsilon > 0$, there exists a constant $C_3 = C_3(m, n, K, \varepsilon) > 0$

$$p_t(x, y) \leq \frac{C_3}{\sqrt{\mu(B_x(\sqrt{t}))\mu(B_y(\sqrt{t}))}} \exp\left(-\frac{d^2(x, y)}{4(1+\varepsilon)t} + \alpha\varepsilon Kt\right). \quad (20)$$

Similarly to [25] (Step 2 of p. 1324), using the Bishop-Gromov relative volume comparison theorem [38, 25], we have

$$\begin{aligned} \mu(B_y(\sqrt{t})) &\leq \mu\left(B_x(d(x, y) + \sqrt{t}) \setminus B_x(d(x, y) - \sqrt{t})\right) \\ &\leq \mu(B_x(\sqrt{t})) \frac{V_{m,K}(d(x, y) + \sqrt{t}) - V_{m,K}(d(x, y) - \sqrt{t})}{V_{m,K}(\sqrt{t})} \\ &\leq \mu(B_x(\sqrt{t})) \frac{V_{m,K}(d(x, y) + \sqrt{t})}{V_{m,K}(\sqrt{t})}, \end{aligned}$$

where $V_{m,K}(r)$ denotes the volume of geodesic balls of radius r in the m -dimensional hyperbolic space form $\mathbb{H}^m(K)$ of constant sectional curvature $K/m - 1$. Using again the Bishop-Gromov volume comparison theorem on $\mathbb{H}^m(K)$, we have

$$\begin{aligned} \frac{\mu(B_y(\sqrt{t}))}{\mu(B_x(\sqrt{t}))} &\leq \frac{V_{m,K}(d(x, y) + \sqrt{t})}{V_{m,K}(\sqrt{t})} \\ &\leq \left(\frac{d(x, y) + \sqrt{t}}{\sqrt{t}}\right)^m \exp[\sqrt{(m-1)K}d(x, y)]. \end{aligned} \quad (21)$$

By (20) and (21), we can obtain the upper bound estimate (18). \square

Proof of Theorem 2.3. By Proposition 4.4, we have

$$\begin{aligned} \frac{p_{t/2}(X_{t/2}, y)}{p_t(x, y)} &\leq C_1(K, m, T) \frac{\mu(B_y(\sqrt{t}))}{\mu(B_y(\sqrt{t/2}))} \exp\left(\frac{d^2(x, y)}{4(1-\varepsilon)t} + \frac{\sqrt{(m-1)K}d(X_{t/2}, y)}{2}\right) \\ &\quad \times \left(\frac{d(X_{t/2}, y) + \sqrt{t}}{\sqrt{t}}\right)^{m/2} \left[\frac{\sqrt{K}d(x, y)}{\sinh \sqrt{K}d(x, y)}\right]^{-\frac{m-1}{2}}. \end{aligned} \quad (22)$$

The Bishop-Gromov volume comparison theorem ([38, 25]) implies

$$\frac{\mu(B_y(\sqrt{t}))}{\mu(B_y(\sqrt{t/2}))} \leq 2^{m/2} \exp(\sqrt{m-1}Kt) \leq 2^{m/2} \exp(\sqrt{m-1}KT). \quad (23)$$

Using (22), (23) and the elementary inequality $x/\sinh x \geq e^{-x}$, we obtain

$$\begin{aligned} \log \left[\frac{p_{t/2}(X_{t/2}, y)}{p_t(x, y)} \right] &\leq C_2(K, m, T) \left[1 + \frac{d(X_{t/2}, y)}{\sqrt{t}} + \frac{d^2(x, y)}{t} + d(X_{t/2}, y) + d(x, y) \right] \\ &\leq C_3(K, m, T) \left[1 + \frac{d^2(X_{t/2}, y)}{t} + \frac{d^2(x, y)}{t} + d(X_{t/2}, y) + d(x, y) \right]. \end{aligned}$$

By Itô's formula, we can prove that (cf. [28])

$$\mathbb{E}_{x,y,T}[d^k(x_{t/2}, y)] \leq (d^k(x, y) + 1)e^{C_4(K,m)t}, \quad \forall t > 0, \quad k = 1, 2.$$

Taking expectation with respect to $\mathbb{E}_{x,y,T}$ in (24), we have

$$\mathbb{E}_{x,y,T} \left[\log \frac{p_{t/2}(X_{t/2}, y)}{p_t(x, y)} \right] \leq C_5(K, m, T) \left[1 + \frac{d^2(x, y) + 1}{t} + d(x, y) \right].$$

By Proposition 4.3, we have

$$\begin{aligned} |\nabla \log p_t(x, y)|^2 &\leq C_6(K, m, T) \left[\frac{1}{t} + \frac{d^2(x, y) + 1}{t^2} + \frac{d(x, y)}{t} \right] \\ &\leq C_7(K, m, T) \left[\frac{d(x, y)}{t} + \frac{1}{\sqrt{t}} \right]^2. \end{aligned}$$

The proof of Theorem 2.3 is completed. \square

Proof of Theorem 2.4. The proof is similarly to Sheu [42] and Hsu [20]. Let $I = \{i_1, \dots, i_N\}$, and denote $H_I = H_{i_1} \dots H_{i_N}$, then by Lemma 4.1 and repeatedly using Itô's formula, we have

$$dH_I J = \langle \nabla^H H_I J, \sqrt{2} db_t \rangle + [\partial_t H_I J + L_{O(M)} H_I J + 2\langle \nabla^H H_I J, \nabla^H J \rangle] dt. \quad (24)$$

Substituting the Hamilton-Jacobi equation (12) into (24), we have

$$dH_I J = \langle \nabla^H H_I J, \sqrt{2} db_t \rangle + (F_I + G_I) dt. \quad (25)$$

where

$$F_I = [L_{O(M)}, H_I] J = [\Delta_{O(M)}, H_I] J - [\nabla^H \phi, H_I] J,$$

and

$$G_I = 2\langle \nabla^H H_I J, \nabla^H J \rangle - H_I |\nabla^H J|^2.$$

By (17) and (22), we have

$$\mathbb{E} \left[\int_0^{T/2} |\nabla^H J(t, U_t)|^2 dt \right] \leq CT \left[\frac{d(x, y)}{T} + \frac{1}{\sqrt{T}} \right]^2, \quad (26)$$

and by Theorem 2.3, we have

$$|H_I J(0, U_0)| \leq C \left[\frac{d(x, y)}{T} + \frac{1}{\sqrt{T}} \right]. \quad (27)$$

Note that

$$[\nabla^H \phi, H_I] J = \sum_{k=1}^n [(H_k \phi) H_k, H_I] J = \sum_{k=1}^n (H_k \phi [H_k, H_I] J - H_I H_k \phi H_k J),$$

and by Cartan's structure theorem, we have

$$[H_i, H_j] = V_{ij}, \quad [H_i, V_{jk}] = \Omega_{jk}^{il} H_l, \quad [V_{ij}, V_{kl}] = c_{ij,kl}^{ab} V_{ab},$$

where V_{ij} are the vertical fields on $O(M)$, Ω is the $o(n)$ -valued curvature form on $O(M)$, and $c_{ij,kl}^{ab}$ are the structure constants of $o(n)$. The assumption of Theorem 2.4 says that the L^∞ -norms $\|\nabla^k \Omega\|_\infty$ are bounded for all $k = 0, 1, \dots, N$, and the L^∞ -norms $\|\nabla^k \phi\|_\infty$ are bounded for $k = 1, \dots, N + 1$. This yields

$$|F_I| \leq C_1 |\nabla^H J| + \dots + C_1 |(\nabla^H)^{|I|} J|, \quad (28)$$

$$|G_I| \leq C_1 |\nabla^H J| + \dots + C_1 |(\nabla^H)^{|I|} J| + C_1 |(\nabla^H)^{|I|} J|^2, \quad (29)$$

where $C_1 > 0$ is a constant depending only on $\|\nabla^i \Omega\|_\infty$ and $\|\nabla^i \phi\|_\infty$, $i = 1, \dots, |I|$. By induction, and using (25), (26), (27), (28) and (29), we can prove that for all I with $2 \leq |I| \leq N$,

$$\mathbb{E} \left[\int_0^{T/2} |H_I J(t, U_t)|^2 dt \right] \leq C_2 \left[\frac{d(x, y)}{T} + \frac{1}{\sqrt{T}} \right]^{2(|I|-1)}, \quad (30)$$

and for all I with $|I| \leq N$,

$$|H_I J(0, U_0)| \leq C_3 \left[\frac{d(x, y)}{T} + \frac{1}{\sqrt{T}} \right]^{|I|}. \quad (31)$$

where C_2 and C_3 are two constants depending on K, m, n, T and C_1 . The proof of Theorem 2.4 is completed. For the detail of the above induction argument for $L = \Delta$ on compact Riemannian manifolds, see Hsu [20]. \square

5 Proofs of Theorem 2.5 and Theorem 2.6

Let

$$H(u) = - \int_M u \log u d\mu.$$

be the Boltzmann-Shannon-Nash entropy of the heat equation of the Witten Laplacian on (M, g, μ) .

To prove Theorem 2.5 and Theorem 2.6, we need the following entropy dissipation formulas for the Witten Laplacian on complete Riemannian manifolds.

Theorem 5.1 ([28]) *Let M be a complete Riemannian manifold, and $\phi \in C^2(M)$. Suppose that there exists a constant $K \geq 0$ such that*

$$\text{Ric} + \nabla^2 \phi \geq -K.$$

Then, for any positive solution to the heat equation

$$\partial_t u = Lu$$

with $u(\cdot, 0) \in L^1(\mu) \cap L^4(d(\cdot, o)d\mu)$, where $o \in M$ is any fixed point, we have

$$\frac{d}{dt} H(u) = - \int_M Lu \log u d\mu. \quad (32)$$

Theorem 5.2 *Let (M, g) be an n -dimensional complete Riemannian manifold. Suppose that $\|\nabla^k \text{Riem}\| \leq C_k$ for some constant $C_k > 0$, $0 \leq k \leq 3$, $\phi \in C^4(M)$ with $\nabla \phi \in C_b^3(M)$, and there exists some $m \geq n$, $K \geq 0$ such that $\text{Ric}_{m,n}(L) \geq -K$. Then, for the fundamental solution of the heat equation $\partial_t u = Lu$, we have*

$$\frac{d^2}{dt^2} H(u) = - \int_M \left(\frac{|Lu|^2}{u} - \left\langle \nabla Lu, \frac{\nabla u}{u} \right\rangle \right) d\mu, \quad (33)$$

and

$$\frac{d^2}{dt^2}H(u) = -2 \int_M (|\nabla^2 \log u|^2 + Ric(L)(\nabla \log u, \nabla \log u))u d\mu. \quad (34)$$

Proof. Based on Theorem 5.1, we can prove (33) by the same argument as used in the proof of Theorem 4.3 in [28]. Integrating by part yields

$$\int_M \frac{|Lu|^2}{u} d\mu = - \int_M \left\langle \nabla \left(\frac{Lu}{u} \right), \frac{\nabla u}{u} \right\rangle u d\mu.$$

Hence

$$\begin{aligned} \frac{d^2}{dt^2}H(u) &= 2 \int_M \left\langle \nabla Lu, \frac{\nabla u}{u} \right\rangle d\mu - \int_M Lu |\nabla \log u|^2 d\mu \\ &= 2 \int_M \left\langle \nabla Lu, \frac{\nabla u}{u} \right\rangle d\mu - \int_M uL |\nabla \log u|^2 d\mu. \end{aligned}$$

By the Bochner formula for the Witten Laplacian, we have (see [28])

$$L|\nabla \log u|^2 = 2\langle \nabla L \log u, \nabla \log u \rangle + 2|\nabla^2 \log u|^2 + 2Ric(L)(\nabla \log u, \nabla \log u)$$

Therefore

$$\begin{aligned} \frac{d^2}{dt^2}H(u) &= 2 \int_M \left\langle \nabla Lu, \frac{\nabla u}{u} \right\rangle d\mu - 2 \int_M \langle \nabla L \log u, \nabla u \rangle d\mu \\ &\quad - 2 \int_M (|\nabla^2 \log u|^2 + Ric(L)(\nabla \log u, \nabla \log u))u d\mu. \end{aligned}$$

Again, integrating by parts yields

$$\int_M \langle \nabla L \log u, \nabla u \rangle d\mu = - \int_M L \log u \cdot Lu d\mu = \int_M \langle \nabla \log u, \nabla Lu \rangle d\mu.$$

Thus

$$\frac{d^2}{dt^2}H(u) = -2 \int_M (|\nabla^2 \log u|^2 + Ric(L)(\nabla \log u, \nabla \log u))u d\mu.$$

The proof of Theorem 5.2 is completed. \square

Remark 5.3 Theorem 5.2 is an improvement of Theorem 4.3 in [28]. The entropy dissipation formulas (32) and (33) play an important role in the study of the convergence to the equilibrium measure of the heat equation $\partial_t u = Lu$ and for the diffusion processes defining by the following SDE on Riemannian manifold (M, g) :

$$dX_t = dW_t - \nabla \phi(X_t) dt, \quad (35)$$

where W_t is a Brownian motion on (M, g) . In the case where M is compact and $\phi \in C^3(M)$, it is well-known that the entropy dissipation formulas (32) and (33) hold. However, as far as we know, even in the non-weighted case, it seems that there is no precisely written result in the literature (except our previous paper [28]) which ensures the entropy dissipation formulas (32) and (33) on general complete non-compact Riemannian manifolds (M, g) . Theorem 5.1 and Theorem 5.2 give us natural geometric conditions on (M, g, ϕ) to ensure (32) and (33).

Proof of Theorem 2.5. We simplify the proof of Theorem 2.3 in [28] as follows. Let

$$H_m(u, t) = - \int_M u \log u d\mu - \frac{m}{2} (1 + \log(4\pi t)).$$

By Theorem 5.1 and Theorem 5.2, we have

$$\frac{d}{dt} H_m(u, t) = - \int_M u L \log u d\mu - \frac{m}{2t} = \int_M \left(|\nabla \log u|^2 - \frac{m}{2t} \right) u d\mu, \quad (36)$$

and

$$\frac{d^2}{dt^2} H_m(u, t) = -2 \int_M (|\nabla^2 \log u|^2 + Ric(L)(\nabla \log u, \nabla \log u)) u d\mu + \frac{m}{2t^2}. \quad (37)$$

By the definition formula (1) of the W -entropy, we have

$$W_m(u, t) = \frac{d}{dt} (t H_m(u, t)) = H_m(u, t) + t \frac{d}{dt} H_m(u, t), \quad (38)$$

and

$$\frac{d}{dt} W_m(u, t) = 2 \frac{d}{dt} H_m(u, t) + t \frac{d^2}{dt^2} H_m(u, t). \quad (39)$$

Substituting (32) and (36) into (38), and using $u = \frac{e^{-f}}{(4\pi t)^{m/2}}$, we obtain

$$\begin{aligned} W_m(u, t) &= - \int_M u \log u d\mu - \frac{m}{2} (1 + \log(4\pi t)) - t \int_M u L \log u d\mu - \frac{m}{2} \\ &= \int_M \left[t |\nabla \log u|^2 - \log u - m - \frac{m}{2} \log(4\pi t) \right] u d\mu \\ &= \int_M [t |\nabla f|^2 + f - m] \frac{e^{-f}}{(4\pi t)^{m/2}} d\mu. \end{aligned}$$

This proves (8) in Theorem 2.5. Substituting (36) and (37) into (39), we obtain

$$\frac{d}{dt} W_m(u, t) = -2 \int_M u L \log u d\mu - 2 \int_M t (|\nabla^2 \log u|^2 + Ric(L)(\nabla \log u, \nabla \log u)) u d\mu - \frac{m}{2t}.$$

Noting that

$$2t |\nabla^2 \log u|^2 + \frac{m}{2t} = 2t \left| \nabla^2 \log u - \frac{g}{2t} \right|^2 + \frac{m-n}{2t} + 2\Delta \log u,$$

and by the integration by part formula

$$\begin{aligned} \int_M \Delta \log u u d\mu &= \int_M (L \log u + \nabla \phi \cdot \nabla \log u) u d\mu \\ &= - \int_M |\nabla \log u|^2 u d\mu + \int_M \nabla \phi \cdot \nabla \log u u d\mu, \end{aligned}$$

we have

$$\begin{aligned} \frac{d}{dt} W_m(u, t) &= -2t \int_M \left| \nabla^2 \log u - \frac{g}{2t} \right|^2 u d\mu + 2 \int_M \nabla \phi \cdot \nabla \log u u d\mu \\ &\quad - 2 \int_M t Ric(L)(\nabla \log u, \nabla \log u) u d\mu - \frac{m-n}{2t}. \end{aligned}$$

By the same argument as in [28], the above identity yields (8). The proof of Theorem 2.5 is completed. \square

Proof of Theorem 2.6. The monotonicity part of Theorem 2.6 follows immediately from Theorem 2.5. The rigidity part of Theorem 2.6 can be proved by the same argument used in Theorem 2.4 in [28]. \square

6 LSI and Lower boundedness of the W -entropy

Following [41], we introduce the best logarithmic Sobolev constant

$$\mu(\tau) := \inf \left\{ W_m(u, \tau) : u = \frac{e^{-f}}{(4\pi\tau)^{m/2}}, \int_M u d\mu = 1 \right\}.$$

In [28], the author proved that $\mu(\tau)$ is always bounded from below on compact Riemannian manifolds M with $\phi \in C^2(M)$. In this section, we prove a family of logarithmic Sobolev inequalities (LSI) on complete Riemannian manifolds with μ -bounded geometry condition.

Proof of Theorem 2.7. Let $v = \sqrt{u}$. Then we can rewrite $W_m(u, \tau)$ as follows

$$W_m(u, \tau) = 4\tau \int_M |\nabla v|^2 d\mu - \int_M v^2 \log v^2 d\mu - \left(m + \frac{m}{2} \log(4\pi\tau) \right).$$

We need to prove, for all $v \in C_0^\infty(M)$ with $\int_M v^2 d\mu = 1$, we have

$$4\tau \int_M |\nabla v|^2 d\mu - \int_M v^2 \log v^2 d\mu - m \left(1 + \frac{1}{2} \log(4\pi\tau) \right) \geq \mu(\tau) > -\infty. \quad (40)$$

Under the conditions of theorem, by Theorem 2.8 in [27], the following L^2 -Sobolev inequality holds: there exists a constant $C_1(m, K, \mu_0) > 0$, depending only on m, K, μ_0 , such that

$$\|f\|_{\frac{2m}{m-2}}^2 \leq C_1(m, K, \mu_0)(\|\nabla f\|_2^2 + \|f\|_2^2), \quad \forall f \in C_0^\infty(M).$$

By [12], the above L^2 -Sobolev inequality implies that, for any $\varepsilon > 0$, there exists a constant $\beta(\varepsilon) > 0$ such that the following logarithmic Sobolev inequality holds

$$\int_M f^2 \log f^2 d\mu \leq \varepsilon \int_M |\nabla f|^2 + \beta(\varepsilon) \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2^2, \quad \forall f \in C_0^\infty(M), \quad (41)$$

where for some constant $C_2(m, K, \mu_0) > 0$ depending only on m, K, μ_0 , we have

$$\beta(\varepsilon) \leq C_2(m, K, \mu_0) - \frac{m}{2} \log \varepsilon.$$

Applying (41) to $f = v \in C_0^\infty(M)$ with $\int_M v^2 d\mu = 1$, and taking $\varepsilon = 4\tau$, we have

$$\int_M v^2 \log v^2 d\mu \leq 4\tau \int_M |\nabla v|^2 - m \left(1 + \frac{1}{2} \log(4\pi\tau) \right) - \mu(\tau),$$

where

$$-\mu(\tau) := \beta(4\tau) + m \left(1 + \frac{1}{2} \log(4\pi\tau) \right),$$

This proves (40) with

$$\mu(\tau) \geq - \left(C_2(m, K, \mu_0) + m + \frac{m}{2} \log(4\pi) \right), \quad \forall \tau > 0.$$

This finishes the proof of Theorem 2.7. \square

7 Analytic proof of Theorem 2.3

In this section, we use Hamilton's Harnack inequality to give an analytic proof of Theorem 2.3. This proof is similar to the one given in [22, 13] for the first order gradient estimate of the logarithmic heat kernel of the Laplace-Beltrami operator on complete Riemannian manifolds with non-negative Ricci curvature.

Analytic proof of Theorem 2.3. Fix $T > 0$, and let $u(t, x)$ be a positive and bounded solution to the heat equation $\partial_t u = Lu$, $t \in (0, t_1)$. Let

$$A := \sup\{u(t, x) : 0 \leq t \leq t_1, x \in M\}.$$

By Hamilton's Harnack inequality (8), we have

$$t|\nabla \log u(t, x)|^2 \leq (1 + 2Kt) \log(A/u(t, x)), \quad \forall (t, x) \in [0, t_1] \times M. \quad (42)$$

Let $s \in (0, T]$, $y \in M$, $t_1 = s/2$ and $u(t, x) = p_{s/2+t}(x, y)$. By (42), (18) and (19), we have

$$\begin{aligned} t|\nabla_x \log p_{s/2+t}(x, y)|^2 &\leq C_{K,m,T}(1 + Kt) \\ &\times \left[1 + \frac{d(x, y)}{\sqrt{t}} + \log \left(\frac{C_1 \mu(B(y, \sqrt{s/2+t}))}{C_2 \mu(B(y, \sqrt{s/2}))} \exp \left(C_3 \frac{d^2(x, y)}{s/2+t} + C_4 d(x, y) \right) \right) \right]. \end{aligned}$$

In particular, taking $t = s/2$ and changing s by t we get

$$\begin{aligned} \frac{t}{2}|\nabla_x \log p_t(x, y)|^2 &\leq C_{K,m,T}(1 + K\frac{t}{2}) \\ &\times \left[1 + \frac{d(x, y)}{\sqrt{t}} + \log \left(\frac{C_1 \mu(B(y, \sqrt{t}))}{C_2 \mu(B(y, \sqrt{t/2}))} \exp \left(C_3 \frac{d^2(x, y)}{t} + C_4 d(x, y) \right) \right) \right]. \end{aligned}$$

By the Bishop-Gromov volume comparison (23), we derive

$$t|\nabla_x \log p_t(x, y)|^2 \leq C_{K,m,T} \left(1 + \frac{d^2(x, y)}{t} + \frac{d(x, y)}{\sqrt{t}} + d(x, y) \right),$$

which yields

$$|\nabla_x \log p_t(x, y)| \leq C_{K,m,T} \left(\frac{d(x, y)}{t} + \frac{1}{\sqrt{t}} \right).$$

This finishes the analytic proof of Theorem 2.3. \square

8 Another probabilistic approach to the Hamilton type Harnack inequality

In this section we develop another probabilistic approach to the Hamilton type Harnack inequality for the Witten Laplacian on complete Riemannian manifolds.

Let X_t be the L -diffusion process on M , and let U_t its horizontal lift to $O(M)$. Let $\mathcal{F}_t = \sigma(X_s, s \leq t)$ be the natural complete σ -field generated by X_t . Let u be a positive solution of the heat equation

$$\partial_t u = L_{O(M)} u.$$

Fix $T > 0$. Let

$$J(t, x) = \log u(T - t, \pi r), \quad \forall r \in O(M), t \in [0, T].$$

Now we introduce the following Doob h -transform for the L -diffusion process X_t on M . Let $\mathbb{P}_{x,T}$ be the law of the conditional L -diffusion process starting from x and with transition density

$$\mathbb{E}_{x,T}(\xi) = \frac{1}{u(T,x)} \mathbb{E}_x[u(T-t, X_t)\xi],$$

where ξ is any bounded \mathcal{F}_t -measurable random variable. By the Girsanov transformation, we have

$$\left. \frac{d\mathbb{P}_{x,T}}{d\mathbb{P}_x} \right|_{\mathcal{F}_t} = \frac{u(T-t, X_t)}{u(T,x)} = \exp\{J(t, U_t) - J(0, U_0)\}, \quad \forall t < T.$$

By Itô's formula, we have

$$dJ(t, U_t) = \frac{\partial J}{\partial t}(t, U_t)dt + \sqrt{2}\langle \nabla^H J(t, U_t), dB_t \rangle + L_{O(M)}J(t, U_t)dt,$$

Note that $J(t, u)$ satisfies the Hamilton-Jacobi equation

$$\frac{\partial J}{\partial t} + L_{O(M)}J + |\nabla^H J|^2 = 0, \quad (43)$$

which yields

$$\left. \frac{d\mathbb{P}_{x,T}}{d\mathbb{P}_x} \right|_{\mathcal{F}_t} = \exp \left[\int_0^t \langle \sqrt{2}\nabla^H J(s, U_s), dB_s \rangle - \int_0^t |\nabla^H J(s, U_s)|^2 ds \right], \quad \forall t < T.$$

It follows, by Girsanov's theorem, that the process

$$b_t = B_t - \sqrt{2} \int_0^t \nabla^H J(s, U_s) ds, \quad s \in [0, T], \quad (44)$$

is a Brownian motion under $\mathbb{P}_{x,T}$. Substituting (44) into (10) and (11), we have the following lemma.

Lemma 8.1 *The conditional L -diffusion process X_t on M starting from x with transition semigroup $\mathbb{P}_{x,T}$, and its horizontal lift U_t , are the solutions of the following Stratonovich SDEs:*

$$\begin{aligned} dU_t &= \sum_{i=1}^n H_i(U_t) \circ \left[\sqrt{2}db_t^i + 2H_i \log u(T-t, U_t)dt \right] - \nabla^H \phi(U_t)dt, \\ dX_t &= U_t \circ \left[\sqrt{2}db_t + 2\nabla^H \log u(T-t, U_t)dt \right] - \nabla \phi(X_t)dt, \end{aligned}$$

where b_t is a standard Brownian motion on \mathbb{R}^n under $\mathbb{P}_{x,T}$.

Proposition 8.2 *For all $0 < t < T$, we have*

$$\mathbb{E}[d|\nabla^H J(t, U_t)|^2] = 2\mathbb{E}[|\nabla^H \nabla^H J|^2 + (\text{Ric} + \nabla^2 \phi)(\nabla^H J, \nabla^H J)]. \quad (45)$$

Proof. The proof is as the same as the one of Proposition 4.2. \square

Theorem 8.3 *Suppose that $\text{Ric} + \nabla^2 \phi \geq -K$. Then for all $T > 0$, we have*

$$|\nabla \log u(T, x)|^2 \leq 2 \left(\frac{1}{T} + K \right) \mathbb{E} \left[\log \frac{u(T/2, X_{T/2})}{u(T, x)} \right]. \quad (46)$$

Proof. The proof is as the same as the one of Proposition 4.3. \square

As a corollary, we can derive the following Hamilton type Harnack inequality for the Witten Laplacian on complete Riemannian manifolds.

Corollary 8.4 *Let u be a bounded and positive solution of the heat equation $\partial_t u = Lu$. Suppose that $\text{Ric} + \nabla^2 \phi \geq -K$. Then for all $T > 0$, we have*

$$|\nabla \log u(T, x)|^2 \leq 2 \left(\frac{1}{T} + K \right) \log (A/u(T, x)), \quad (47)$$

where

$$A := \sup\{u(t, x) : x \in M, t \in [0, T]\}.$$

Proof. The Hamilton type Harnack inequality (47) follows from (46) in Theorem 8.3. \square

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